XXIII. *On the Conditions for the existence of Three Equal Roots, or of Two Pairs of Equal Roots, of a Binary Quartic or Quintic.* *By* A. Cayley, *F.R.S.*

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In considering the conditions for the existence of given systems of equalities between the roots of an equation, we obtain some very interesting examples of the composition of relations. A relation is either onefold, expressed by a single equation $U=0$, or it is, say k-fold, expressed by a system of k or more equations. Of course, as regards onefold relations, the theory of the composition is well known: the relation $UV=0$ is a relation compounded of the relations $U=0$, $V=0$; that is, it is a relation satisfied if, and not satisfied unless one or the other of the two component relations is satisfied. The like notion of composition applies to relations in general; viz., the compound relation is a relation satisfied if, and not satisfied unless one or the other of the two component relations is satisfied. I purposely refrain at present from any further discussion of the theory of composition. I say that the conditions for the existence of given systems of equalities between the roots of an equation furnish instances of such composition; in fact, if we express that the function $(*\between x, y)^n$, and its first-derived function in regard to x, or, what is the same thing, that the first-derived functions in regard to x, y respectively, have a common quadric factor, we obtain between the coefficients a certain twofold relation, which implies either that the equation $(*\between x, y)^n=0$ has three equal roots, or else that it has two pairs of equal roots; that is, the relation in question is satisfied if, and it is not satisfied unless there is satisfied either the relation for the existence of three equal roots, or else the relation for the existence of two pairs of equal roots; or the relation for the existence of the quadric factor is compounded of the last-mentioned two relations. The relation for the quadric factor, for any value whatever of n, is at once seen to be expressible by means of an oblong matrix, giving rise to a series of determinants which are each to be put $=0$; the relation for three equal roots and that for two pairs of equal roots, in the particular cases $n=4$ and $n=5$, are given in my "Memoir on the Conditions for the existence of given Systems of Equalities between the roots of an Equation," Phil. Trans. vol. cxlvii. (1857), pp. 727–731; and I propose in the present Memoir to exhibit, for the cases in question $n=4$ and $n=5$, the connexion between the compound relation for the quadric factor with the component relations for the three equal roots and for the two pairs of equal roots respectively.

Article Nos. 1 to 8, the Quartic.

1. For the quartic function

$$(a, b, c, d, e\between x, y)^4,$$

the condition for three equal roots, or, say, for a root system 31, is that the quadrinvariant and the cubinvariant each of them vanish, viz. we must have

$$\mathrm{I}=ae-4bd+3c^2=0,$$
$$\mathrm{J}=ace-ad^2-b^2e+2bcd-c^3=0.$$

2. The condition for two pairs of equal roots, or for a root system 22, is that the cubicovariant vanishes identically, viz. representing this by

$$(\mathrm{A, B, 5C, 10D, 5E, F, G}\between x, y)^6=0,$$

we must have

$$\begin{aligned}
\mathrm{A}&=\ a^2d-3abc+2b^3 &&=0,\\
\mathrm{B}&=\ a^2e+2abd-9ac^2+6b^2c &&=0,\\
\mathrm{C}&=\ abe-3acd+2b^2d &&=0,\\
\mathrm{D}&=-ad^2+b^2e &&=0,\\
\mathrm{E}&=-ade+3bce-2bd^2 &&=0,\\
\mathrm{F}&=-ae^2-2bde+9c^2e-6cd^2 &&=0,\\
\mathrm{G}&=-be^2+3cde-2d^3 &&=0.
\end{aligned}$$

3. But the condition for the common quadric factor is

$$\left\|\begin{array}{ccccc} & a, & 3b, & 3c, & d \\ & b, & 3c, & 3d, & e \\ a, & 3b, & 3c, & d & \\ b, & 3c, & 3d, & e & \end{array}\right\|=0,$$

and the determinants formed out of this matrix must therefore vanish for (I, J)=0, and also for (A, B, C, D, E, F, G)=0, that is, the determinants in question must be syzygetically related to the functions (I, J), and also to the functions (A, B, C, D, E, F, G).

4. The values of the determinants are—

1234=3×	1235=3×	1245=	1345=3×	2345=3×
$+1\ a^2ce$	$-1\ a^2de$	$-1\ a^2e^2$	$-1\ abe^2$	$+1\ ace^2$
$-3\ a^2d^2$	$+4\ abce$	$+2\ abde$	$+4\ acde$	$-1\ ad^2e$
$-1\ ab^2e$	$+1\ abd^2$	$+9\ ac^2e$	$-3\ ad^3$	$-3\ b^2e^2$
$+14\ abcd$	$-3\ ac^2d$	$-9\ acd^2$	$+1\ b^2de$	$+14\ bcde$
$-9\ ac^3$	$-3\ b^3e$	$-9\ b^2ce$	$-3\ bc^2e$	$-8\ bd^3$
$-8\ b^3d$	$+2\ b^2cd$	$+8\ b^2d^2$	$+2\ bcd^2$	$-9\ c^3e$
$+6\ b^2c^2$				$+6\ c^2d^2$

5. The syzygetic relation with (I, J) is given by means of the identical equation

$$\left|\begin{array}{ccccc} y^4, & -4xy^3, & 6x^2y^2, & -4x^3y, & x^4 \\ & a\ , & 3b\ , & 3c\ , & d \\ & b\ , & 3c\ , & 3d\ , & e \\ a\ , & 3b\ , & 3c\ , & d\ , & \\ b\ , & 3c\ , & 3d\ , & e\ , & \end{array}\right|=-6\mathrm{I}.\tilde{\mathrm{H}}\mathrm{U}+9\mathrm{J}.\mathrm{U},$$

or, as this may be written,

$$(1234,\ 1235,\ 1245,\ 1345,\ 2345 \between x, y)^4 = -6\mathrm{I}.\tilde{\mathrm{H}}\mathrm{U} + 9\mathrm{J}.\mathrm{U},$$

where $\tilde{\mathrm{H}}\mathrm{U}$ is the Hessian of U,

$$= \left(\begin{array}{|c|c|c|c|c|} ac & 2\ ad & ae & 2\ be & ce \\ -b^2 & -2\ bc & +2\ bd & -2\ cd & -d^2 \\ & & -3\ c^2 & & \end{array} \between x,\ y\right)^4.$$

6. That is, we have

$$\begin{array}{rl} 1234 = & (\ ac - b^2 \quad\quad\ ,\ \ a \between -6\mathrm{I},\ 9\mathrm{J}), \\ 4.\ 1235 = & (2ad - 2bc \quad\quad ,\ 4b \between -6\mathrm{I},\ 9\mathrm{J}), \\ 6.\ 1245 = & (\ ae + 2bd - 3c^2,\ 6c \between -6\mathrm{I},\ 9\mathrm{J}), \\ 4.\ 1345 = & (2be - 2cd \quad\quad ,\ 4d \between -6\mathrm{I},\ 9\mathrm{J}), \\ 2345 = & (\ ce - d^2 \quad\quad\ ,\ \ e \between -6\mathrm{I},\ 9\mathrm{J}). \end{array}$$

7. The determinants thus vanish if $(\mathrm{I}, \mathrm{J}) = 0$, that is, for the root system 31; they will also vanish without this being so, if only

$$\left(\frac{3\mathrm{J}}{2\mathrm{I}} = \right) \frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e};$$

and we may omit the first member $\left(\frac{3\mathrm{J}}{2\mathrm{I}} = \right)$, since if the remaining terms are equal to each other they will also be $= \frac{3\mathrm{J}}{2\mathrm{I}}$. The equations may then be written

$$\left\| \begin{array}{ccccc} ac - b^2, & ad - bc, & ae + 2bd - 3c^2, & be - cd, & ce - d^2 \\ a\ \ , & 2b\ \ , & 6c\ \ , & 2d\ \ , & e \end{array} \right\| = 0,$$

and the ten equations of this system reduce themselves (as it is very easy to show) to the seven equations

$$(\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D},\ \mathrm{E},\ \mathrm{F},\ \mathrm{G}) = 0,$$

which, as above mentioned, are the conditions for the root system 22.

8. It may be added that we have

	A	B	C	D	E	F	G
$\frac{1}{3}.1234=$		c	$-4b$	$+3a$			
$\frac{1}{3}.1235=$			c	$-3b$	$+\ a$		
$0\ =$		d	$-3c$		$+\ a$		
$1245=$		$-e$	$+4d$	$-3c$			
$0\ =$		$-e$		$+6c$		$-a$	
$0\ =$			$-\ d$	$+3c$	$-\ b$		
$\frac{1}{3}.1345=$			$-\ e$	$+3d$	$-\ c$		
$0\ =$			$-\ e$		$+3c$	$-b$	
$\frac{1}{3}.2345=$				$-3e$	$+4d$	$-c$	

where it is to be noticed that the four equations having the left-hand side $=0$, give

B : C : D : E : F proportional to the determinants of the matrix

$$\left|\begin{array}{ccccc} d, & -3c, & , & a & \\ -e, & ., & 6c, & ., & -a \\ & -d\,, & 3c, & -\,b, & \\ & -e\,, & ., & +3c, & -b \end{array}\right| ;$$

the determinants in question contain each the factor c, and omitting this factor, the system shows that B, C, D, E, F are proportional to their before-mentioned actual values.

Article Nos. 9 to 15, the Quintic.

9. For the quintic function

$$(a,\ b,\ c,\ d,\ e,\ f \between x,\ y)^5,$$

the condition of a root system 41 is that the covariant, Table No. 14, shall vanish, viz. we must have

$$\begin{aligned} A&=2(ae-4bd+3c^2)=0, \\ B&=\ \ af-3be+2cd=0, \\ C&=2(bf-4ce+3d^2)=0. \end{aligned}$$

10. The condition of a root system 32 is that the following covariant, viz.

$$3\,(\text{No. }13)^2(\text{No. }14)-25(\text{No. }15)^2,$$

shall vanish, where

No. 13 $= (a,\ b,\ c,\ d,\ e, f \between x,\ y)^5$, the quintic itself.

$$\text{No. 14} = \left(\begin{array}{r|r|r} ae & af & bf \\ -4\ bd & -3\ be & -4\ ce \\ +3\ c^2 & +2\ cd & +3\ d^2 \end{array} \between x,\ y\right)^2.$$

$$\text{No. 15} = \left(\begin{array}{r|r|r|r|r|r|r} ac & 3\ ad & 3\ ae & af & 3\ bf & 3\ cf & df \\ -b^2 & -3\ bc & +3\ bd & +7\ be & +3\ ce & -3\ de & -e^2 \\ & & -6\ c^2 & -8\ cd & -6\ d^2 & & \end{array} \between x,\ y\right)^6.$$

11. The developed expression of the foregoing function is as follows:—

	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	
(	$+3\ a^3e$	$+3\ a^3f$	$+33\ a^2bf$	$+10\ a^2cf$	$-90\ a^2df$	$-114\ a^2ef$	$-19\ a^2f^2$	$-114\ abf^2$	$-90\ acf^2$	$+10\ adf^2$	$+33\ aef^2$	$+3\ af^3$	$+3\ bf^3$	$)(x, y)^{12}$
	$-12\ a^2bd$	$+21\ a^2be$	$-102\ a^2ce$	$-390\ a^2de$	$-195\ a^2e^2$	$-264\ abdf$	$-608\ abef$	$-264\ acef$	$+360\ adef$	$+155\ ae^2f$	$-102\ bdf^2$	$+21\ bef^2$	$-12\ cef^2$	
	$-16\ a^2c^2$	$-144\ a^2cd$	$-216\ a^2d^2$	$+155\ ab^2f$	$+360\ abcf$	$-990\ abe^2$	$+537\ acdf$	$+468\ ad^2f$	$+225\ ae^3$	$-390\ bcf^2$	$+135\ be^2f$	$-144\ cdf^2$	$-16\ d^2f^2$	
	$+50\ ab^2c$	$+30\ ab^2d$	$+135\ ab^2e$	$+100\ abce$	$-1500\ abde$	$+468\ ac^2f$	$-245\ ace^2$	$+900\ ade^2$	$-195\ b^2f^2$	$+100\ bdef$	$-216\ c^2f^2$	$+30\ ce^2f$	$+50\ de^2f$	
	$-25\ b^4$	$+240\ abc^2$	$+120\ abcd$	$-600\ abd^2$	$+900\ ac^2e$	$+1320\ acde$	$+1740\ ad^2e$	$-990\ b^2ef$	$-1500\ bcef$	$+125\ be^3$	$+120\ cdef$	$+240\ d^2ef$	$-25\ e^4$	
		$-150\ b^3c$	$+480\ ac^3$	$+1600\ ac^2d$	$+1800\ acd^2$	$+1080\ ad^3$	$-245\ b^2df$	$+1320\ bcdf$	$+900\ bd^2f$	$-600\ c^2ef$	$-150\ ce^3$	$-150\ de^3$		
			$-150\ b^3d$	$+125\ b^3e$	$+225\ b^3f$	$+900\ b^2cf$	$-1700\ b^2e^2$	$-2700\ bce^2$	$+1800\ c^2df$	$+1600\ cd^2f$	$+480\ d^3f$			
			$-300\ b^2c^2$	$-1000\ b^2cd$	$-1500\ b^2d^2$	$-2700\ b^2de$	$+1740\ bc^2f$	$+900\ bd^2e$	$-1500\ c^2e^2$	$-1000\ cde^2$	$-300\ d^2e^2$			
						$+900\ bc^2e$	$-2000\ bcde$	$+1080\ c^3f$						
						$-600\ bcd^2$	$+600\ bd^3$	$-600\ c^2de$						
							$+600\ c^3e$							
							$-400\ c^2d^2$							

12. The conditions for the common quadric factor are

$$\left\| \begin{array}{cccccc} & a, & 4b, & 6c, & 4d, & e \\ a, & 4b, & 6c, & 4d, & e & \\ & b, & 4c, & 6d, & 4e, & f \\ b, & 4c, & 6d, & 4e, & f & \end{array} \right\| = 0,$$

the several determinants whereof are given in Table No. 27 of my "Third Memoir on Quantics," Philosophical Transactions, vol. cxlvi. (1856), pp. 627–647.

13. These determinants must therefore vanish, for (A, B, C)=0, and also for $(\mathfrak{A}, \mathfrak{B}, \ldots \mathfrak{L}, \mathfrak{M})=0$, that is, they must be syzygetically connected with (A, B, C), and also with $(\mathfrak{A}, \mathfrak{B}, \ldots \mathfrak{L}, \mathfrak{M})$. The relation to (A, B, C) is in fact given in the Table appended to Table No. 27, viz. this is

	C×	+B×	+A×
1234=	$+6\ a^2$	$-12\ ab$	$+16\ ac-10\ b^2$
1235=	$+6\ ab$	$-2\ ac-10\ b^2$	$+6\ ad$
1236=	$-2\ ac+8\ b^2$	$+6\ ad-18\ bc$	$-2\ df+8\ e^2$
1245=	$+18\ ac$	$-6\ ad-30\ bc$	$+8\ ae+10\ bd$
1246=	$+12\ bc$	$+4\ ae-4\ bd-24\ c^2$	$+4\ be+8\ cd$
1345=	$+24\ ad$	$-8\ ae-40\ bd$	$+4\ af+20\ be$
1256=	$-1\ ae+4\ bd+3\ c^2$	$+1\ af+5\ be-18\ cd$	$-1\ bf+4\ ce+3\ d^2$
2345=	$+20\ ae+40\ bd-30\ c^2$	$-80\ be+20\ cd$	$+20\ bf+40\ ce-30\ d^2$
1346=	$+4\ ae+8\ bd+6\ c^2$	$-36\ cd$	$+4\ bf+8\ ce+6\ d^2$
2346=	$+4\ af+20\ be$	$-8\ bf-4\ ce$	$+24\ cf$
1356=	$+4\ be+8\ cd$	$+4\ bf-4\ ce-24\ d^2$	$+12\ de$
2356=	$+8\ bf+10\ ce$	$-6\ cf-30\ de$	$+18\ df$
1456=	$+6\ ce$	$+6\ cf-18\ de$	$-2\ df+8\ e^2$
2456=	$+6\ cf$	$-2\ df-10\ e^2$	$+6\ ef$
3456=	$+16\ df-10\ e^2$	$-12\ ef$	$+6\ f^2$

14. Between the expressions $\mathfrak{A}$, $\mathfrak{B}$, &c., and 1234, 1235, &c., there exist relations the form of which is indicated by the following Table:

$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$	$\mathfrak{D}$	$\mathfrak{E}$	$\mathfrak{F}$	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{I}$	$\mathfrak{J}$	$\mathfrak{K}$	$\mathfrak{L}$	$\mathfrak{M}$	1234	1235	1236	1245	1246	1345	1256	2345	1346	2346	1356	2356	1456	2456	3456
c	b	a											a														
d	c	b	a										b	a													
e	d	c	b	a									c	b	a	a											
f	e	d	c	b	a								d	c	b	b	a	a									
	f	e	d	c	b	a							e	d	c	c	b	b	a	a	a						
		f	e	d	c	b	a						f	e	d	d	c	c	b	b	b	a	a				
			f	e	d	c	b	a						f	e	e	d	d	c	c	c	b	b	a	a		
				f	e	d	c	b	a						f	f	e	e	d	d	d	c	c	b	b	a	
					f	e	d	c	b	a							f	f	e	e	e	d	d	c	c	b	a
						f	e	d	c	b	a								f	f	f	e	e	d	d	c	b
							f	e	d	c	b	a										f	f	e	e	d	c
								f	e	d	c	b												f	f	e	d
									f	e	d	c														f	e
										f	e	d															f

viz. these relations are of the form

$$(\,)c\mathfrak{A}+(\,)b\mathfrak{B}+(\,)a\mathfrak{C} \qquad +(\,)a.1234 \qquad\qquad =0,$$

$$(\,)d\mathfrak{A}+(\,)c\mathfrak{B}+(\,)b\mathfrak{C}+(\,)a\mathfrak{D} \qquad +(\,)b.1234+(\,)a.1235=0,$$

&c.

where the brackets ( ) denote numerical coefficients, determinate as to their ratios.

15. Assuming the existence of these relations, we have for the determination of the numerical coefficients in each relation a set of linear equations, which are shown by the following Tables, viz. referring to the Table headed $c\mathfrak{A}$, $b\mathfrak{B}$, $a\mathfrak{C}$, $a.1234$, if the multipliers of terms respectively be A, B, C, X, then the Table denotes the system of linear equations

$$\begin{aligned} &0\ \mathrm{A} + 3\,\mathrm{B} + 33\,\mathrm{C} + 0\,\mathrm{X}=0,\\ &3\ \mathrm{A} + 0\,\mathrm{B} - 102\,\mathrm{C} - 16\,\mathrm{X}=0,\\ &\&c., \end{aligned}$$

that is, nine equations to be satisfied by the ratios of the coefficients A, B, C, X, and which are in fact satisfied by the values at the foot of the Table, viz.

$$\mathrm{A}:\mathrm{B}:\mathrm{C}:\mathrm{X}=66:-11:+1:+6.$$

There would be in all fourteen Tables, but as those for the second seven would be at once deducible by symmetry from the first seven, I have only written down the seven Tables; the solutions for the first and second Tables were obtained without difficulty, but that for the third Table was so laborious to calculate, and contains such extraordinarily high numbers, that I did not proceed with the calculation, and it is accordingly only the first, second, and third Tables which have at the foot of them respectively the solutions of the linear equations.

16. The results given by these three Tables are, of course,

$$66c\mathfrak{A}- 11b\mathfrak{B}+ 1a\mathfrak{C}+6a.1234=0,$$

$$330d\mathfrak{A}+110c\mathfrak{B}-55b\mathfrak{C}+9a\mathfrak{D}-105\ a.1235=0,$$

$$\begin{aligned} &+266478575\ e\mathfrak{A}\\ &-617359490\ d\mathfrak{B}\\ &+144200810\ c\mathfrak{C}\\ &+\quad 9656911\ b\mathfrak{D}\\ &+\quad 9090785\ a\mathfrak{E}\\ &-721004050\ c\,.1234\\ &+\ 90914175\ b\,.1235\\ &-160758675\ a\,.1245\\ &+\ 11559295\ a\,.1236=0. \end{aligned}$$

It is to be noticed that the nine coefficients of this last equation were obtained from, and that they actually satisfy, a system of fourteen linear equations; so that the correctness of the result is hereby verified.

17. The seven Tables are

First Table.

	$c\mathfrak{A}$	$b\mathfrak{B}$	$a\mathfrak{C}$	$a.1234$
a^3bf		+ 3	+ 33	
a^3ce	+ 3		−102	− 16
a^3d^2			−216	+ 36
a^2be^2		+ 21	+135	+ 16
a^2bcd	−12	−144	+120	−152
a^2c^3	−16		+480	+ 96
ab^3d		+ 30	−150	+ 80
ab^2c^2	+50	+240	−300	− 60
b^4c	−25	−150		
	+66	−11	+1	+ 6

Second Table.

	$d\mathfrak{A}$	$c\mathfrak{B}$	$b\mathfrak{C}$	$a\mathfrak{D}$	$a.1235$	$b.1234$
a^3cf		+ 3		+ 10	− 4	
a^3de	+ 3			− 390	+24	
a^2b^2f			+ 33	+ 155	+ 4	
a^2bce		+ 21	−102	+ 100	−84	− 16
a^2bd^2	−12		−216	− 600	−24	+ 36
a^2c^2d	−16	−144		+1600	+64	+ 16
ab^3e			+135	+ 125	+60	
ab^2cd	+50	+ 30	+120	−1000	−40	−152
abc^3		+240	+480			+ 96
b^4d	−25		−150			+ 80
b^3c^2		−150	−300			− 60
	+330	+110	−55	+9	−105	0

Third Table.

	$e\mathfrak{A}$	$d\mathfrak{B}$	$c\mathfrak{C}$	$b\mathfrak{D}$	$a\mathfrak{E}$	$c.1234$	$b.1235$	$a.1245$	$a.1236$
a^3df		+ 3			− 90			− 6	+ 6
a^3e^2	+ 3				− 195			+16	
a^2bcf			+ 33	+ 10	+ 360		− 4	+ 6	−22
a^2bde	−12	+ 21		− 390	−1500		+24	−26	− 6
a^2c^2e	−16		−102		+ 900	− 16		−96	+16
a^2cd^2		−144	−216		+1800	+ 36		+96	
ab^3f				+ 155	+ 225		+ 4		+16
ab^2ce	+50		+135	+ 100		+ 16	−84	+90	−10
ab^2d^2		+ 30		− 600	−1500		−24	−80	
abc^2d		+240	+120	+1600		−152	+64		
ac^4			+480			+ 96			
b^4e	−25			+ 125			+60		
b^3cd		−150	−150	−1000		+ 80	−40		
b^3c^3			−300			− 60			
	+266478575	−617359490	+144200810	+9656911	+9090785	−721004050	+90914175	−160758675	+11559295

Fourth Table.

	$f\mathfrak{A}$	$e\mathfrak{B}$	$d\mathfrak{C}$	$c\mathfrak{D}$	$b\mathfrak{E}$	$a\mathfrak{F}$	$d.1234$	$c.1235$	$b.1236$	$b.1245$	$a.1246$	$a.1345$
a^3ef	+ 3	+ 3				− 114					+ 4	
a^2bdf	− 12		+ 33		− 90	− 264			+ 6	− 6	− 4	− 24
a^2be^2		+ 21			− 195	− 990				+ 16	− 4	+ 64
a^2c^2f	− 16			+ 10		+ 468		− 4			− 24	+ 24
a^2cde		− 144	− 102	− 390		+ 1320	− 16	+ 24			+ 24	− 208
a^2d^3			− 216			+ 1080	+ 36					+ 144
ab^2cf	+ 50			+ 155	+ 360	+ 900		+ 4	− 22	+ 6	+ 24	
ab^2de		+ 30	+ 135		− 1500	− 2700	+ 16		− 6	− 26	− 20	− 40
abc^2e		+ 240		+ 100	+ 900	+ 900		− 84	+ 16	− 96		+ 60
$abcd^2$			+ 120	− 600	+ 1800	− 600	− 152	− 24		+ 96		− 40
ac^3d			+ 480	+ 1600			+ 96	+ 64				
b^4f	− 24				+ 225				+ 16			
b^3ce		− 150		+ 125				+ 60	− 10	+ 90		
b^3d^2			− 150		− 1500		+ 80			− 80		
b^2c^2d			− 300	− 1000			− 60	− 40				

Fifth Table.

	$f\mathfrak{B}$	$e\mathfrak{C}$	$d\mathfrak{D}$	$c\mathfrak{E}$	$b\mathfrak{F}$	$a\mathfrak{G}$	$e.1234$	$d.1235$	$c.1236$	$c.1245$	$b.1246$	$b.1345$	$a.1256$	$a.2345$	$a.1346$
a^3f^2	+ 3					− 19							+ 1		
a^2bef	+ 21	+ 33			− 114	− 608					+ 4		− 2		+ 16
a^2cdf	− 144		+ 10	− 90		+ 537		− 4	+ 6	− 6			− 16	+ 20	− 36
a^2ce^2		− 102		− 195		− 245	− 16			+ 16			+ 16	− 80	− 16
a^2d^2e		− 216	− 390			+ 1740	+ 36	+ 24					+ 16	+ 60	+ 36
ab^2df	+ 30		+ 155		− 264	− 245		+ 4			− 4	− 24	− 15	− 80	− 16
ab^2e^2		+ 135			− 990	− 1700	+ 16				− 4	+ 64		+ 240	
abc^2f	+ 240			+ 360	+ 468	+ 1740			− 22	+ 6	− 24	+ 24		+ 60	+ 36
$abcde$		+ 120	+ 100	− 1500	+ 1320	− 2000	− 152	− 84	− 6	− 26	+ 24	− 208		− 860	− 20
abd^3			− 600		+ 1080	+ 600		− 24				+ 144		+ 960	
ac^3e		+ 480		+ 900		+ 600	+ 96		+ 16	− 96				+ 960	
ac^2d^2			+ 1600	+ 1800		− 400		+ 64		+ 96				− 320	
b^3cf	− 150			+ 225	+ 900				+ 16		+ 24				
b^3de		− 150	+ 125		− 2700		+ 80	+ 60			− 20	− 40			
b^2c^2e		− 300			+ 900		− 60		− 10	+ 90		+ 60			
b^2cd^2			− 1000	− 1500	− 600			− 40		− 80		− 40			

Sixth Table.

	$f\mathfrak{C}$	$e\mathfrak{D}$	$d\mathfrak{E}$	$c\mathfrak{F}$	$b\mathfrak{G}$	$a.\mathfrak{H}$	$f.1234$	$e.1235$	$d.1236$	$d.1245$	$c.1246$	$c.1345$	$b.1256$	$b.2345$	$b.1346$	$a.2346$	$a.1356$
a^2bf^2	+33				−19	−114							+1				+4
a^2cef	−102	+10		−114		−264	−16	−4			+4					−24	−4
a^2d^2f	−216		−90			+468	+36		+6	−6						+24	−24
a^2de^2		−390	−195			+900		+24		+16							+24
ab^2ef	+135	+155			−698	−990	+16	+4					−2		+16	+64	−4
$abcdf$	+120		+360	−264	+537	+1320	−152		−22	+6	−4	−24	−16	+20	−36	−208	+24
$abce^2$		+100		−990	−245	−2700		−84			−4	+64	+16	−80	−16	−40	−20
abd^2e		−600	−1500		+1740	+900		−24	−6	−26			+16	+60	+36	+60	
ac^3f	+480			+468		+1080	+96				−24	+24				+144	
ac^2de		+1600	+900	+1320		−600		+64	+16	−96	+24	−208				−40	
acd^3			+1800	+1080						+96		+144					
b^3df	−150		+225		−245		+80		+16				−80	−80	−16		
b^3e^2		+125			−1700			+60						+240			
b^2c^2f	−300			+900	+1740		−60				+24			+60	+36		
b^2cde		−1000		−2700	−2000			−40	−10	+90	−20	−40		−860	−20		
b^2d^3			−1500		+600					−80				+960			
bc^3e				+900	+600							+60		+960			
bc^2d^2				−600	−400							−40		−320			

Seventh Table.

	$f\mathfrak{D}$	$e\mathfrak{E}$	$d\mathfrak{F}$	$c\mathfrak{G}$	$b\mathfrak{H}$	$a\mathfrak{I}$	$f.1235$	$e.1236$	$e.1245$	$d.1246$	$d.1345$	$c.1256$	$c.2345$	$c.1346$	$b.2346$	$b.1356$	$a.2356$	$a.1456$
a^2cf^2	+10			−19		−90	−4					+1					−6	+6
a^2def	−390	−90	−114			+360	+24	+6	−6	+4							+6	−22
a^2e^3		−195				+225			+16									+16
ab^2f^2	+155				−114	−195	+4									+4	+16	
$abcef$	+100	+360		−608	−264	−1500	−84	−22	+6			−2		+16	−24	−4	−26	−6
abd^2f	−600		−264		+468	+900	−24			−4	−24				+24	−24	−96	+16
$abde^2$		−1500	−990		+900			−6	−26	−4	+64					+24	+90	−10
ac^2df	+1600		+468	+537		+1800	+64			−24	+24	−16	+20	−36			+96	
ac^2e^2		+900		−245		−1500		+16	−96			+16	−80	−16			−80	
acd^2e		+1800	+1320	+1740					+96	+24	−208	+16	+60	+36				
ad^4			+1080								+144							
b^3ef	+125	+225			−990		+60	+16							+64	−4		
b^2cdf	−1000		+900	−245	+1320		−40			+24		−15	−80	−16	−208	+24		
b^2ce^2				−1700	−2700			−10	+90				+240		−40	−20		
b^2d^2e		−1500	−2700		+900				−80	−20	−40				+60			
bc^3f				+1740	+1080								+60	+36	+144			
bc^2de			+900	−2000	−600						+60		−860	−20	−40			
bcd^3			−600	+600							−40		+960					
c^4e				+600									+960					
c^3d^2				−400									−320					

And the remaining seven Tables might of course be deduced from these by writing (f, e, d, c, b, a) instead of (a, b, c, d, e, f), and making the corresponding alterations in the top line of each Table.

18. The equations $\mathfrak{A}=0, \mathfrak{B}=0, \ldots, \mathfrak{M}=0$ consequently establish between the fifteen functions 1234, 1235, ... 3456 a system of fourteen equations, viz. the first and last three of these are

$$\begin{array}{r}
1234=0,\\
1235=0,\\
-160758675 \,.\, 1245\\
+\ 11559295 \,.\, 1236=0,\\
\cdot\\
\cdot\\
\cdot\\
+\ 11559295 \,.\, 1456\\
-160758675 \,.\, 2356=0,\\
2456=0,\\
3456=0.
\end{array}$$

To complete the proof that in virtue of the equations $\mathfrak{A}=0$, $\mathfrak{B}=0, \ldots, \mathfrak{M}=0$ all the fifteen functions 1234, 1235, ... 3456 vanish, it is necessary to make use of the identical relations subsisting between these quantities 1234, &c.; thus we have

$$a \,.\, 1345+4b \,.\, 1245+6c \,.\, 1235+4d \,.\, 1234=0,$$

$$b \,.\, 1345+4c \,.\, 1245+6d \,.\, 1235+4e \,.\, 1234=0,$$

which, in virtue of the above equations $1234=0$ and $1235=0$, become

$$a \,.\, 1345+4b \,.\, 1245=0,$$

$$b \,.\, 1345+4c \,.\, 1245=0,$$

giving (unless indeed $ac-b^2=0$) $1245=0$, $1345=0$; the equation $1245=0$ then reduces the third of the above equations to $1236=0$, and so on until it is shown that the fifteen quantities all vanish.